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Mild-slope approximation for long waves generated by short waves

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Abstract. The mild-slope approximation has become a popular basis for calculating infinitesimal surface waves on slowly varying depth. It is less restrictive hence more advantageous than the ray and parabolic approximations for describing diffraction and refraction by bathymetry and/or by complex coastlines. Since its computation involves only two horizontal coordinates, the mild-slope equation is also numerically less demanding than the solution of fully three-dimensional equations for a horizontal area with sides much greater than the typical wavelength. By consideration of nonlinear effects of the second order, the mild-slope approximation for long waves over slowly varying depth is derived here, in order to provide a convenient basis for predicting long-period resonance in a large harbor by short-period wind waves.

Keywords: harbor oscillations, nonlinear resonance, mild-slope approximation, short/long waves interactions

1. Introduction

In many harbors of horizontal dimensions $O(1-10\,\mathrm{km})$ oscillations of periods of $O(1-10\,\mathrm{min})$ have often been recorded. When one of these modes is resonated, costly damages to ship fenders and mooring lines and hazards to loading and unloading are possible. Expensive remedies by adding or modifying breakwaters may not be effective unless the mechanism of resonance can be properly predicted. As an example, to eliminate recorded slow oscillations in Pier J and in the Naval Basin at the Port of Long Beach, California, renovations are being proceeded on the basis of linearized theory of resonance. In both laboratory and numerical models, the incident waves are assumed to have the same frequency as that of the resonant mode. This assumption is, however, questionable since the incident waves are not of tsunami origin and are caused by distant wind. Energy in the long-period part of the sea spectrum is the nonlinear consequence of the short wind waves of periods in the $5 \sim 15$-second range. In principle, the long waves cannot be treated separately from the short waves.

There exist a few theories treating long-period harbor oscillations by groups of short-period wind waves. Bowers [1] considered two narrow channels in series with the outer channel being wider and the waves are long-crested in both channels. Mei and Agnon [2] gave an analytical theory for a narrow bay perpendicular to a long and straight coast; the sea depth being constant everywhere. Wu and Liu [3] considered a rectangular harbor with two breakwaters along a straight coast, also for constant depth. The physical picture described in all these theories is that periodic groups of narrow-banded incident short waves are accompanied by long-period set-down waves propagating at the group velocity of the short carrier waves. Upon reflection by the coast and scattering by the harbor entrance, free long waves of much longer period, unforced by radiation stresses, are also generated in and outside the harbor. These free long waves, which have the characteristic velocity $\sqrt{gh}$, can be resonated inside the harbor to significant amplitude. Other related theories on long waves by short wave groups have been published earlier by Mei and Benmoussa [4] for refraction over one-dimensional bathymetry, Agnon and Mei [5] for diffraction and radiation by a two-dimensional floating cylinder, Agnon and Mei [6] for diffraction by an infinitely long shelf, and Zhou and Liu [7] for scattering by a circular island on a horizontal seabed. There is yet no general theory for effective calculation of the combined effects of bathymetry and lateral boundaries (coastline, breakwaters).

Since caustics can occur over a mildly sloping seabed, a good theoretical model must be able to account for both refraction and diffraction. For resonance by long-period tsunamis the linearized long-wave equation can be treated effectively by the hybrid-element method of Chen and Mei [8]. For short wind waves over slowly varying bathymetry, the ray approximation, which involves one-dimensional computations, is useful only if caustics are not present. When either caustics or breakwaters are present, the mild-slope equation (MSE) is a better tool since it has the important advantage of accounting for both refraction and diffraction. From the computational standpoint MSE reduces the mathematical problem from three- to two-space dimensions, and can also be solved by the hybrid-element method, as demonstrated by Houston [9]. Various extensions of the mild-slope equation have been made by several authors in recent years (Booij [10] for waves on a current; Radder [11] for narrow frequency band, Kirby [12], Chamberlain and Porter [13] and Porter and Staziker [14] for not-so-mild bed slope).

To treat harbor oscillations due to narrow-banded short waves which are both refracted and diffracted by slowly varying bathymetry, it is desirable to extend the mild-slope approximation to long waves as well. Let h be the typical depth k, A, ω and $\Delta\omega$ be the typical wave number, amplitude, frequency, and frequency band width, respectively. Then $\varepsilon = kA$ represents the typical wave steepness and $\mu = \Delta\omega/\omega$ the rate of modulation. In this paper we shall assume these two ratios to be small and comparable

$$\varepsilon = O(\mu) \ll 1. \tag{1.1}$$

In addition the spatial rate of depth variation with a wavelength is also small

$$\frac{\nabla h}{kh} = O(\mu) = O(\varepsilon) \ll 1. \tag{1.2}$$

We shall obtain the mild-slope approximation for both the short and long waves up to the second order $O(\varepsilon^2)$, $O(\varepsilon\mu)$ and for the ranges of $\mu\omega t = O(1)$ and $(\mu kx, \mu ky) = O(1)$. The resulting equations can be used as the basis for numerical modelling of harbor problems by the hybrid-element method. Steps for application to harbor oscillations are sketched, and numerical implementation will be reported in the future.

2. The governing equations

For an incompressible and inviscid fluid the velocity potential is governed by Laplace's equation in the fluid

$$\frac{\partial^2\phi}{\partial z^2} + \nabla^2\phi = 0 \quad -h(x, y) < z < \zeta(x, y, t), \tag{2.1}$$

where ∇ denotes the horizontal gradient

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \quad \text{and} \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2.2}$$

On the seabed there can be no normal flux

$$\frac{\partial \phi}{\partial z} = -\nabla\phi \cdot \nabla h. \tag{2.3}$$

On the free surface $z = \zeta(x, y, t)$, we assume that the atmospheric pressure is a constant. The kinematic and dynamic conditions can be combined to give

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = -\frac{\partial}{\partial t}(\mathbf{u})^2 - \tfrac{1}{2}\mathbf{u}\cdot\nabla\mathbf{u}^2, \tag{2.4}$$

where $\mathbf{u} = (\partial\phi/\partial x, \partial\phi/\partial y, \partial\phi/\partial z)$. The free-surface displacement is related to ϕ by the Bernoulli equation

$$\zeta = -\frac{1}{g}\frac{\partial \phi}{\partial t} + \frac{1}{2g}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right]. \tag{2.5}$$

Anticipating that the free-surface displacement corresponding to the slow oscillations be of $O(\varepsilon^2)$, or $O(\varepsilon\mu)$ we expect the potential to have zeroth and first harmonics at $O(\varepsilon)$, and second harmonics at $O(\varepsilon^2)$, *i.e.*,

$$\phi = \varepsilon(\phi_0 + \phi_1\,\mathrm{e}^{-i\omega t} + *) + \varepsilon^2(\phi_2\,\mathrm{e}^{-2i\omega t} + *) + O(\varepsilon^3), \tag{2.6}$$

where $*$ denotes the complex conjugate of the preceding term, and $\phi_n = \phi_n(x, y, z, t')$, $n = 0, 1, 2, \ldots$ with $t' = \mu t$. All terms ϕ_n are of order unity and include terms of higher orders in ε and μ. In principle, multiple-scale spatial coordinates can also be introduced to describe the first and slow variations in horizontal directions. Note that $\varepsilon\phi_0$, which is of order $O(\varepsilon)$, varies with the time slowly and affects the pressure field at the second order through $\varepsilon\mu\partial\phi_0/\partial t'$ in the Bernoulli equation.

To examine long waves of frequency $O(\varepsilon\omega)$ or $O(\mu\omega)$ it is necessary to keep the term $\varepsilon\mu^2\partial^2\phi_0/\partial t'^2$ which must arise from (2.4); hence it is necessary to keep some high-order terms including $O(\varepsilon\mu^2)$ and $O(\mu\varepsilon^2)$. Specifically, after Taylor expansion about $z = 0$, quadratic terms in (2.4) and (2.5) must be kept, *i.e.*,

$$\begin{aligned}\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} \;=\; & \frac{\partial}{\partial t}\left\{-\frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + \frac{1}{g}\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t\partial z}\right\}\\ & -\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial y}\right) + O(\varepsilon^3).\end{aligned} \tag{2.7}$$

All terms of the right will contribute to terms that vary in time only slowly and of the order $O(\varepsilon^2\mu)$. On the other hand, cubic terms, which give rise to odd harmonics only, will not contribute to the long wave at $O(\varepsilon^3)$.

3. Harmonic decomposition

Substituting (2.6) in (2.1) and (2.3), we get after separating the time-harmonics

$$\frac{\partial^2 \phi_n}{\partial z^2} + \nabla^2 \phi_n = 0, \quad -h < z < \zeta, \tag{3.1}$$

where h is a slowly varying function of x and y. Similarly, from the seabed boundary conditions we get from various harmonics

$$\frac{\partial \phi_n}{\partial z} = -\nabla \phi_n \cdot \nabla h \quad n = 0, 1, 2, \ldots \ . \tag{3.2}$$

On the free surface the boundary conditions involve time derivatives. For narrow-banded waves it is convenient to use the fast and slow time variables t and $t' = \mu t$ so that

$$\frac{\partial^2 \phi}{\partial t^2} \Rightarrow \frac{\partial^2 \phi}{\partial t^2} + 2\mu \frac{\partial^2 \phi}{\partial t \partial t'} + \mu^2 \frac{\partial^2 \phi}{\partial t'^2}. \tag{3.3}$$

Substituting (2.6) in (2.7) and separating the harmonics, we get, by keeping terms of orders $O(\varepsilon)$, $O(\varepsilon\mu)$ and $O(\varepsilon\mu^2)$, and then dividing the entire boundary condition by ε, the following zeroth harmonic

$$\frac{\partial \phi_o}{\partial z} = -\frac{\mu^2}{g} \frac{\partial^2 \phi_0}{\partial t'^2} + F_0, \quad z = 0, \tag{3.4}$$

where

$$\begin{aligned} F_0 \;=\; & -\frac{\varepsilon\mu}{g} \left\{ \left(\frac{\partial \phi_0}{\partial x} \frac{\partial^2 \phi_0}{\partial x \partial t'} + \frac{\partial \phi_0}{\partial y} \frac{\partial^2 \phi_0}{\partial y \partial t'} + \frac{\partial \phi_0}{\partial z} \frac{\partial^2 \phi_0}{\partial z \partial t'} \right) \right. \\ & \qquad - \frac{\partial}{\partial t'} \left(\left| \frac{\partial \phi_1}{\partial x} \right|^2 + \left| \frac{\partial \phi_1}{\partial y} \right|^2 + \left| \frac{\partial \phi_1}{\partial z} \right|^2 \right) \\ & \qquad \left. + \frac{\omega^2}{g} \left(\phi_1 \frac{\partial^2 \phi_1^*}{\partial z \partial t'} + \frac{\partial \phi_1}{\partial t'} \frac{\partial \phi_1^*}{\partial z} + * \right) \right\} \\ & -\frac{\varepsilon}{g} \left\{ \frac{\partial}{\partial x} \left[\left(-i\omega\phi_1 + \mu \frac{\partial \phi_1}{\partial t'} \right) \frac{\partial \phi_1^*}{\partial x} + * \right] \right. \\ & \qquad \left. + \frac{\partial}{\partial y} \left[\left(-i\omega\phi_1 + \mu \frac{\partial \phi_1}{\partial t'} \right) \frac{\partial \phi_1^*}{\partial y} + * \right] \right\} \\ & -\frac{\varepsilon\mu}{g} \left\{ \frac{\partial}{\partial x} \left(\frac{\partial \phi_0}{\partial t'} \frac{\partial \phi_0}{\partial x} + * \right) + \frac{\partial}{\partial y} \left(\frac{\partial \phi_0}{\partial t'} \frac{\partial \phi_0}{\partial y} + * \right) \right\}. \end{aligned} \tag{3.5}$$

All terms on the right-hand side are evaluated at $z = 0$.

For the first harmonic, it suffices to keep terms of order $O(\varepsilon)$, $O(\varepsilon^2)$ and $O(\varepsilon\mu)$ with the result

$$\frac{\partial \phi_1}{\partial z} = \frac{\omega^2}{g} \phi_1 + F_1, \quad z = 0, \tag{3.6}$$

where

$$F_1 = \varepsilon \frac{2i\omega}{g} \nabla\phi_0 \cdot \nabla\phi_1 + \mu 2i\omega \frac{\partial \phi_1}{\partial t'}, \quad z = 0. \tag{3.7}$$

Finally, for the second harmonic it is only necessary to keep $O(\varepsilon^2)$ and $O(\varepsilon\mu)$ terms, yielding

$$\frac{\partial \phi_2}{\partial z} = \frac{4\omega^2}{g}\phi_2 + F_2, \quad z = 0, \tag{3.8}$$

where

$$\begin{aligned} F_2 \;=\; & \frac{i\omega}{g}\left\{\left(\frac{\partial\phi_1}{\partial x}\right)^2 + \left(\frac{\partial\phi_1}{\partial y}\right)^2 + \left(\frac{\partial\phi_1}{\partial z}\right)^2 + \frac{2\omega^2}{g}\phi_1\frac{\partial\phi_1}{\partial z}\right. \\ & \left. + \frac{\partial}{\partial x}\left(\phi_1\frac{\partial\phi_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi_1\frac{\partial\phi_1}{\partial y}\right)\right\}, \quad z = 0. \end{aligned} \tag{3.9}$$

In view of the boundary conditions (3.2) and (3.4), $\partial\phi_o/\partial z = 0$ on both $z = 0$ and h to the leading order. This suggests, subject to *a posteriori* check for consistency, that ϕ_0 is independent of the fast coordinates x, y, z to the leading order, *i.e.*,

$$\phi_0 = \phi_{00}\left(x', y', t'\right) + \varepsilon\phi_{01}\left(x, y, z, x', y', t'\right) + O(\varepsilon^2). \tag{3.10}$$

Thus

$$\nabla\phi_0 = \mu\nabla'\phi_{00} + \varepsilon\nabla\phi_{01} + O(\varepsilon^2, \varepsilon\mu, \mu^2), \tag{3.11}$$

where

$$\nabla' = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}\right),$$

and F_0 and F_1 may then be simplified to

$$\begin{aligned} F_0 \;=\; & -\frac{\varepsilon\mu}{g}\left\{-\frac{\partial}{\partial t'}\left(\left|\frac{\partial\phi_1}{\partial x}\right|^2 + \left|\frac{\partial\phi_1}{\partial y}\right|^2 + \left|\frac{\partial\phi_1}{\partial z}\right|^2\right) + \frac{\omega^2}{g}\left(\phi_1\frac{\partial^2\phi_1^*}{\partial z\partial t'} + \frac{\partial\phi_1}{\partial t'}\frac{\partial\phi^*}{\partial z} + *\right)\right\} \\ & -\frac{\varepsilon}{g}\left\{\frac{\partial}{\partial x}\left[\left(-i\omega\phi_1 + \mu\frac{\partial\phi_1}{\partial t'}\right)\frac{\partial\phi_1^*}{\partial x} + *\right] + \frac{\partial}{\partial y}\left[\left(-i\omega\phi_1 + \mu\frac{\partial\phi_1}{\partial t'}\right)\frac{\partial\phi_1^*}{\partial y} + *\right]\right\} \end{aligned} \tag{3.12}$$

and

$$F_1 = 2i\mu\omega\frac{\partial\phi_1}{\partial t'}. \tag{3.13}$$

4. Mild-slope approximation

For monochromatic surface waves of infinitesimal amplitude, the approach of Smith and Spinks [15] for deriving the mild-slope equation is as follows. They first assume a solution for ϕ_1 in the form which is valid for locally constant depth, and then take a weighted vertical average of the three-dimensional equation via Green's formula. Here the three potentials ϕ_0, ϕ_1 and ϕ_2 are governed by similar linear boundary-value problems, hence their approach can be applied to all.

4.1. First harmonic

For convenience we first recall the procedure of Smith and Spinks, and introduce

$$f(x, y, z) = \frac{\cosh k(z+h)}{\cosh kh} \tag{4.1}$$

with

$$\omega^2 = gk \tanh kh. \tag{4.2}$$

Note that f is the homogeneous solution to the boundary value problem,

$$\frac{\partial^2 f}{\partial z^2} - k^2 f = 0, \quad -h < z < 0, \tag{4.3}$$

$$\frac{\partial f}{\partial z} - \frac{\omega^2}{g} f = 0, \quad z = 0, \tag{4.4}$$

$$\frac{\partial f}{\partial z} = 0, \quad z = -h. \tag{4.5}$$

Applying Green's formula to f and ϕ_1 we get

$$\int_{-h}^{0} \left(f \frac{\partial^2 \phi_1}{\partial z^2} - \phi_1 \frac{\partial^2 f}{\partial z^2} \right) \mathrm{d}z = \left[f \frac{\partial \phi_1}{\partial z} - \phi_1 \frac{\partial f}{\partial z} \right]_h^0 . \tag{4.6}$$

By using the conditions governing f: (4.3) to (4.5)and ϕ_1: (3.1), (3.2) and (3.6), we can easily prove that

$$\int_{-h}^{0} \nabla^2 \phi_1 \, \mathrm{d}z + \int_{-h}^{0} k^2 f \phi_1 \, \mathrm{d}z + f \, [\nabla \phi_1]_{-h} \cdot \nabla h = - \left[f F_1 \right]_{z=0} , \tag{4.7}$$

We now assume

$$\phi_1 = -\frac{ig}{\omega} \eta(x, y) f(x, y, z) \tag{4.8}$$

and substitute this, (4.1) and (3.13) in (4.6). Keeping only terms of $O(\mu^0)$ and $O(\mu)$, we get

$$\nabla \cdot \left(C C_g \nabla \eta \right) + \omega^2 \frac{C_g}{C} \eta + 2i\omega\mu \frac{\partial \eta}{\partial t'} = 0. \tag{4.9}$$

Where C and C_g denote the phase and group velocities, respectively

$$C = \frac{\omega}{k}, \qquad C_g = \frac{C}{2}\left(1 + \frac{2kh}{kh\,\sinh}\right). \tag{4.10}$$

Without the third term on the left, this is the original mild-slope equation (MSE) for monochromatic waves. The additional term is consistent with the extension by Radder [11] based on a Hamiltonian approach.

4.2. Zeroth harmonic

Integration of (3.1) for $n = 0$ from $z = -h$ to $z = 0$ gives

$$\int_{-h}^{0} \frac{\partial^2 \phi_0}{\partial z^2}\,\mathrm{d}z = \left[\frac{\partial \phi_0}{\partial z}\right]_0 - \left[\frac{\partial \phi_0}{\partial z}\right]_{-h}. \tag{4.11}$$

This amounts to the application of Green formula to ϕ_0 and 1. Employing the boundary conditions on $z = 0$ and h, (3.2) and (3.4), we get

$$\mu^2 \frac{\partial^2 \phi_0^2}{\partial t'^2} - g\nabla \cdot (h\nabla\phi_0) = gF_0. \tag{4.12}$$

This is the mild-slope approximation for the zeroth time-harmonic which contains fast and slow variation in space. To gain further insight and to check the consistency of the assumption (3.10), we use multiple-scale coordinates in space, $x_i' = \mu x_i$, in order to separate terms of different orders in (4.12). At the order $O(\varepsilon)$ we get

$$\varepsilon\{-g\nabla \cdot (h\nabla\phi_{01})\} = -\varepsilon\left[\frac{\partial}{\partial x}\left(-i\omega\phi_1\frac{\partial \phi_1^*}{\partial x} + *\right) + \frac{\partial}{\partial y}\left(-i\omega\phi_1\frac{\partial \phi_1^*}{\partial y} + *\right)\right]$$

or, after canceling ε,

$$g\nabla \cdot (h\nabla\phi_{01}) = \nabla \cdot (-i\omega\phi_1\nabla\phi_1^* + *), \tag{4.13}$$

which governs the fast spatial variations of ϕ_{01}. At the next order $O(\mu^2, \varepsilon\mu)$ we get

$$\begin{aligned}
&\mu^2\left[\frac{\partial \phi_{00}}{\partial t'^2} - g\nabla' \cdot \left(h\nabla'\phi_{00}\right)\right] - \varepsilon\mu g\left[\nabla \cdot \left(h\nabla'\phi_{01}\right) + \nabla' \cdot (h\nabla\phi_{01})\right] \\
&\quad = -\varepsilon\mu\left\{\frac{\partial}{\partial t'}\left(\left|\frac{\partial \phi_1}{\partial x}\right|^2 + \left|\frac{\partial \phi_1}{\partial y}\right|^2 + \left|\frac{\partial \phi_1}{\partial z}\right|^2\right) + \frac{\omega^2}{g}\left(\phi_1\frac{\partial^2 \phi_1^*}{\partial z\partial t'} + \frac{\partial \phi_1}{\partial t'}\frac{\partial \phi_1^*}{\partial z} + *\right)\right\} \\
&\qquad -\varepsilon\mu\{\nabla' \cdot (-i\omega\phi_1\nabla\phi_1^* + *) + \nabla \cdot (-i\omega\phi_1\nabla'\phi_1^* + *) \\
&\qquad +\nabla \cdot (\phi_{1t'}\nabla\phi_1^* + *)\},
\end{aligned} \tag{4.14}$$

which governs the slow variation of ϕ_{00} and ϕ_{01}, *i.e.*, of ϕ_0. Thus (3.10) gives a consistent perturbation scheme.

4.3. SECOND HARMONIC

Let us define f_2 by the homogeneous boundary-value problem

$$\frac{\partial^2 f_2}{\partial z^2} - k_2^2 f_2 = 0, \quad -1 < z < 0, \tag{4.15}$$

$$\frac{\partial f_2}{\partial z} - \frac{4\omega^2}{g} f_2 = 0, \quad z = 0, \tag{4.16}$$

$$\frac{\partial f_2}{\partial z} = 0, \quad z = -h, \tag{4.17}$$

then

$$f_2(x, y, z) = \frac{\cosh\ k_2(z+h)}{\cosh\ k_2 h} \tag{4.18}$$

with

$$(2\omega)^2 = g k_2 \tanh k_2 h. \tag{4.19}$$

Applying Green's formula to f_2 and ϕ_2, we get

$$\int_{-h}^{0} \left(f_2 \nabla^2 \phi_2 + k_2^2 f_2 \phi_2 \right) \mathrm{d}z + f_2 \left[\nabla \phi_2 \right]_{-h} \cdot \nabla h = -F_2. \tag{4.20}$$

The boundary term on the left is $O(\mu)$ smaller than the rest and can be discarded. Upon letting

$$\phi_2 = -\frac{ig\sigma}{2\omega} f_2, \tag{4.21}$$

we obtain an equation governing the fast variation of σ

$$\nabla^2 \sigma + k_2^2 \sigma = G_2, \tag{4.22}$$

where

$$G_2 = \frac{g}{2\omega} C_{g_2} \left\{ 2 \left(\frac{\partial \eta}{\partial x} \right)^2 + 2 \left(\frac{\partial \eta}{\partial y} \right)^2 + \frac{3\omega^4}{g^2} \eta^2 + \eta \nabla^2 \eta \right\} \tag{4.23}$$

and

$$C_{g_2} = \frac{1}{2} \frac{2\omega}{k_2} \left(1 + \frac{2 k_2 h}{\sinh k_2 h} \right). \tag{4.24}$$

We note that only the short-scale variation of σ is of concern; h can be treated as locally constant.

4.4. SUMMARY

Equations (4.9), (4.12) and (4.22) constitute the mild-slope approximation for short and long waves up to the second order in ε or μ. For a practical coastal geometry and bathymetry, numerical means are necessary. As the first step, one can employ the hybrid-element method of Chen and Mei [8] to calculate the short-wave potential ϕ_1. This requires the discretization of a large domain with a resolution fine enough over the short wavelength, in order to give the fast and slow variation of ϕ_1. Afterwards the forcing functions for ϕ_0 can be calculated from (4.9), with F_0 defined by (3.12). We remark that, in the special case where ϕ_1 is essentially periodic over the fast coordinates (x, y), as in the case for pure refraction, the ray approximation applies, then ϕ_{01} is also periodic over the fast scales. The slow evolution of the long-wave part ϕ_{00} can be obtained from the spatial average of (4.14). Subsequently, the variation of ϕ_{01} can be solved. In general, however, ϕ_1 involves diffraction and is not at all spatially periodic; such a separation by homogenization is not possible. It is then necessary to solve (4.12) for ϕ_0 as a single equation. Thus, it is necessary to discretize a large domain with a resolution fine enough over the short wavelength, but this is no more demanding than what is needed for solving (4.9) for the short waves.

When ϕ_0, ϕ_1 and ϕ_2 are solved, the velocity potential ϕ is given by (2.7), and the free-surface displacement by

$$\zeta = -\frac{1}{g}\left(\frac{\partial \phi}{\partial t} + \zeta \frac{\partial^2 \phi}{\partial t \partial z}\right) + \frac{1}{2g}\left(\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right) + O(\varepsilon^3), \tag{4.25}$$

where all terms on the right are evaluated on $z = 0$. This formula can be deomposed into harmonics

$$\zeta = \varepsilon(\zeta_1 \,\mathrm{e}^{-i\omega t} + *) + \varepsilon^2(\zeta_0 + \zeta_2 \,\mathrm{e}^{-2i\omega t} + *) \tag{4.26}$$

with

$$\zeta_1 = \frac{i\omega}{g}\phi_1, \tag{4.27}$$

$$\zeta_0 = \frac{1}{g}\left\{-\frac{\partial \phi_0}{\partial t'} + \frac{\omega^2}{g^2}\left(\phi_1 \frac{\partial \phi_1^*}{\partial z} + *\right) + \frac{1}{2}\left(\left|\frac{\partial \phi_1}{\partial x}\right|^2 + \left|\frac{\partial \phi_1}{\partial y}\right|^2 + \left|\frac{\partial \phi_1}{\partial z}\right|^2\right)\right\}, \tag{4.28}$$

$$\zeta_2 = \frac{1}{g}\left\{-\frac{\omega^2}{g^2}\phi_1 \frac{\partial \phi_1}{\partial z} + \frac{1}{2}\left(\frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial y}\right)^2 + \left(\frac{\partial \phi_1}{\partial z}\right)^2\right\}. \tag{4.29}$$

5. Comparison with existing results

It is well-known for a monochromatic train of short waves that the mild-slope equation (MSE) is quite robust despite the fact the derivation is only based on weighted depth averaging. In particular, in two limiting cases where diffraction can be treated: (a) finite wavelength and horizontal seabed and (b) shallow water waves over varying depth, MSE agrees with the known equations derivable by the rigorous perturbation scheme. On the other hand, for pure refraction of monochromatic waves on slowly varying depth, the relation between MSE and the ray approximation has been discussed by Jonsson [16] , Kirby [17] and Dingemans [18] (see Liu, [19], for a review). It is easy to check that for refracting waves with a narrow frequency band, (4.9) leads to the well-known conservation equation for wave action. Specifically, we let

$$\eta = A\, \mathrm{e}^{iS/\mu}, \tag{5.1}$$

where $A = A(x', y', t')$ is the wave amplitude and $S(x', y')/\mu$ is the wave phase with

$$\mathbf{k} = \nabla' S. \tag{5.2}$$

It follows by using the definition $\omega = kC$ at order $O(\mu^0)$ and by collecting all $O(\mu)$ terms in (4.14) that we have

$$2\frac{\partial A}{\partial t'} + \nabla' \cdot (\mathbf{C}_g A) + \mathbf{C}_g \cdot \nabla' A = 0. \tag{5.3}$$

where $\mathbf{C}_g = C_g \mathbf{k}/k$. Multiplying the preceding equation by A^* and adding the result to its own complex conjugate, we get

$$\frac{\partial |A|^2}{\partial t'} + \nabla' \cdot (\mathbf{C}_g |A|^2) = 0, \tag{5.4}$$

which agrees with the law of wave-action conservation for narrow-banded waves.

We now turn to the long waves induced nonlinearly by narrow-banded short waves. By systematic multiple-scale analysis, several results for the following limiting cases are known: (a) a horizontal seabed (Agnon and Mei, [5]) and (b) pure refraction over slowly varying seabed (Mei and Benmoussa, [4]). Indeed, if the depth is constant and short waves are horizontally periodic (*e.g.*, progressive or partially standing waves), the spatial average of (4.14) over a horizontal period gives (5.10) of Agnon and Mei [5]. In the limiting case where ϕ_1 is a train of progressive waves, the long-wave potential is known to be governed by

$$\frac{\partial^2 \phi_0}{\partial t'^2} - gh\nabla'^2 \phi_0 = \frac{2\omega^3}{k} \frac{\cosh^2 kh}{\sinh^2 kh} \frac{\partial\, |A|^2}{\partial x'} - \frac{\omega^2}{\sinh^2 kh} \frac{\partial\, |A|^2}{\partial t'} \tag{5.5}$$

(see Mei [20], p. 613). By straightforward calculations, the right-hand side of (4.12) according to (3.12) is precisely the right-hand side of (5.5), except for a factor $\varepsilon\mu$. Since all the forcing terms vary slowly in x', the left-hand side of (4.12) becomes

$$\mu^2 \left(\frac{\partial^2 \phi_0}{\partial t'^2} - gh\nabla'^2 \phi_0 \right). \tag{5.6}$$

Removing the ordering parameters, we recover Equation (5.5). Thus (4.12) is the generalization of (5.5) for a mild-bottom slope and general short waves experiencing diffraction and refraction.

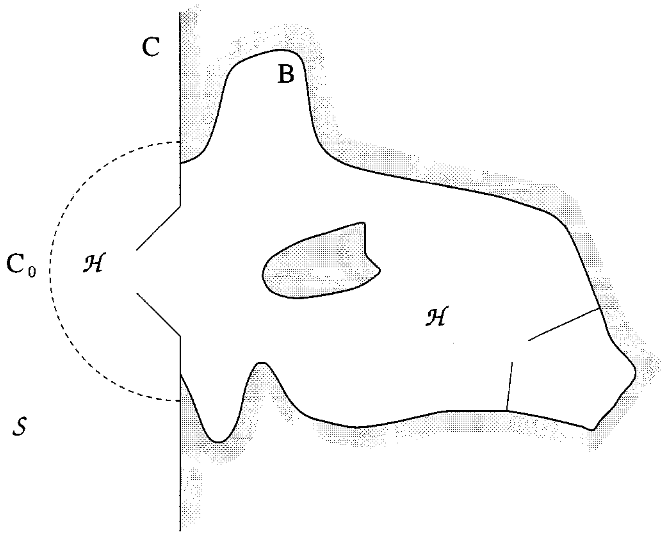

Figure 1. Harbor with complex bathymetry and coastline.

6. Harbor attacked by periodically modulated wind waves

As an illustration, we consider a large harbor as sketched in Figure 1. Let C_o be a semicircle of radius large enough to include all topographical irregularities. The harbor, denoted by $\mathcal{H}$ is defined to be the fluid region bounded on the right by C_o. The boundary of the harbor is designated by B. The sea, denoted by $\mathcal{S}$, is defined to be the fluid region to the left (outside) of C_o and the straight coastline C. For simplicity the sea depth in $\mathcal{S}$ is assumed to be constant h_o.

Let the incident short (wind) waves arrive at the incidence angle θ_o, then

$$\eta^I = A^I \exp(ik_o x \cos\theta_o + ik_o y \sin\theta_o), \tag{6.1}$$

where k_o is the real root of (4.2) for $h = h_o$. Assume for simplicity of demonstration that the incident wave envelope is sinusoidal,

$$A^I = A \exp(Kx' \cos\theta_o + Ky' \sin\theta_o - i\Omega t') + *, \tag{6.2}$$

where

$$\Omega = C_g K. \tag{6.3}$$

If we leave the effects of the harbor for later correction, the reflected short waves from the straight coast is given by

$$\eta^R = A^R \exp(-ik_o x \cos\theta_o + ik_o y \sin\theta_o) \tag{6.4}$$

with the envelope

$$A^R = A \exp(-Kx' \cos\theta_o + Ky' \sin\theta_o - i\Omega t') + *. \tag{6.5}$$

6.1. The harbor

Within the harbor $\mathscr{H}$ we can write the short-wave displacement as

$$\eta = \eta^{-} e^{-i\Omega t'} + \eta^{+} e^{i\Omega t'}. \tag{6.6}$$

From (4.9) we get

$$\nabla \cdot (C C_g \nabla \eta^{\pm}) + \left(\frac{\omega^2 C_g}{C} \mp 2\mu\omega\Omega \right) \eta^{\pm} = 0, \tag{6.7}$$

which is an elliptic equation. Let the shoreline B be reflective; the normal velocity must vanish so that

$$\frac{\partial \eta^{\pm}}{\partial n} = 0, \quad \text{on } B. \tag{6.8}$$

The forcing term F_0 in (4.12) (see (3.12)) involves zeroth and second harmonics in Ω with respect to the long-period oscillations. The former forces steady set-up or set-down, or steady current and is of no concern to harbor resonance; the corresponding response will be denoted by $\langle \phi_0 \rangle$. We shall be interested only in the sinusoidal forcing and its response, hence

$$\phi_0 = \phi_0^{-} e^{-2i\Omega t'} + \phi_0^{+} e^{2i\Omega t'} + \langle \phi_0 \rangle. \tag{6.9}$$

It follows from (4.12) that

$$\nabla \cdot (h \nabla \phi_0^{\pm}) + \frac{4\mu^2\Omega^2}{g} \phi_0^{\pm} = -F_0^{\pm}, \tag{6.10}$$

where $F_0^{\pm}$ is defined by

$$F_0 = F_0^{-} e^{-2i\Omega t'} + F_0^{+} e^{2i\Omega t'} + \langle F_0 \rangle. \tag{6.11}$$

The boundary conditions on B is

$$\frac{\partial \phi_0^{\pm}}{\partial n} = 0 \quad \text{on } B. \tag{6.12}$$

The second harmonic displacement σ can be similarly decomposed and will be omitted.

6.2. The sea

The short-wave displacement can be written as

$$\eta = \eta^{I} + \eta^{R} + \eta^{0}, \tag{6.13}$$

where η^0 denotes the outgoing short waves radiated from the harbor

$$\eta^0 = \sum_{n=0}^{\infty} H_n^{(1)}(k_o r) \sin\theta A_n \, e^{2i(Kr' - \Omega t')} + * \tag{6.14}$$

with $H_n^{(1)}(k_o r)$ denoting the Hankel function of the first kind and $\theta = \pm\pi/2$ along the straight coast. Note that the solution satisfies the no-flux condition along the coast C. The constant complex coefficients A_n are yet unknown.

In the sea $\mathcal{S}$, ϕ_0 consists of slowly oscillating parts as well as nonoscillating parts

$$\phi_0 = \phi_0^I + \phi_0^R + \phi_0^F + \langle\phi_0\rangle. \tag{6.15}$$

The oscillating parts consists of long waves ϕ_0^I, ϕ_0^R bound to the incident and the reflected short waves and the free long waves ϕ_0^F. The bound long waves can be obtained in terms of the short wave and takes the form

$$\begin{aligned}\begin{pmatrix}\phi_0^I\\ \phi_0^R\end{pmatrix} &= \begin{pmatrix}\phi_0^{I-}\mathrm{e}^{-2i\Omega t'} + \phi_0^{I+}\mathrm{e}^{2i\Omega t'}\\ \phi_0^{R-}\mathrm{e}^{-2i\Omega t'} + \phi_0^{R+}\mathrm{e}^{2i\Omega t'}\end{pmatrix}\\ &= \begin{pmatrix}C^I\,\mathrm{e}^{2iKx'\cos\theta_o}\\ C^R\,\mathrm{e}^{-2iKx'\cos\theta_o}\end{pmatrix}\mathrm{e}^{2i(Ky'\sin\theta_o-\Omega t')} + *,\end{aligned} \tag{6.16}$$

where C^I and C^R are constant coefficients in terms of the short incident and reflected waves. Because of the radial attenuation as $(kr)^{-1/2}$, the scattered short waves η^S do not induce bound long waves, as pointed out by Zhou and Liu [7]. The free long waves represented by ϕ_0^F have the characteristic velocity $\sqrt{gh_o}$ and must be of the form:

$$\phi_0^F = \phi_0^{F-}\mathrm{e}^{-2i\Omega t'} + \phi_0^{F+}\mathrm{e}^{2i\Omega t'} = \sum_{n=0}^{\infty} D_m H_n^{(1)}(2K_o r')\sin n\theta\,\mathrm{e}^{-2i\Omega t'} + *, \tag{6.17}$$

where D_m are unknown expansion coefficients, and

$$K_o = \frac{\Omega}{\sqrt{gh_o}}. \tag{6.18}$$

Along the semicircle C_o we must require the continuity of pressure and flux, therefore

$$\{\phi_0\}_{\mathcal{H}} = \{\phi_0\}_{\mathcal{S}}, \tag{6.19}$$

$$\left\{\frac{\partial\phi_0}{\partial r}\right\}_{\mathcal{H}} = \left\{\frac{\partial\phi_0}{\partial r}\right\}_{\mathcal{S}}. \tag{6.20}$$

6.3. Numerical strategy

The hybrid-element method of Chen and Mei [8] can be applied to the boundary-value problems for both the short wave $\eta^{\pm}$ and the long wave $\phi_0^{\pm}$. The idea is to replace each boundary-value problem by a variational principle which involves an area integral over the harbor $\mathcal{H}$ and a line integral along C_o. It suffices to demonstrate the strategy for $\phi_0^{\pm}$. Briefly the potential $\phi_0^{\pm}$ in $\mathcal{H}$ is approximated by finite elements. The stationary functional is then a quadratic form of the unknown nodal coefficients and of the unknown expansion coefficients $\phi_0^{F\pm}$ in $\mathcal{S}$. Extremization of the functional with respect to each coefficient leads to matrix equations which can be solved.

We now give the stationary functional J for the oscillatory part of the zeroth harmonic potential proportional to the time factor $\mathrm{e}^{\pm 2i\Omega t'}$. Let the superscripts $\mathcal{H}$ and $\mathcal{S}$ distinguish the potentials in the harbor and in the sea

$$\phi_0^{\mathcal{H}} = \phi_0^{\pm}, \qquad \phi_0^{\mathcal{S}} = \phi_0^{F\pm} + \phi_0^{I\pm} + \phi_0^{R\pm}. \tag{6.21}$$

and let the overhead bar denote the difference between the total potential and the parts due to the bound long waves

$$\bar{\phi}_0^{\mathcal{S}} = \phi_0^{F\pm}, \qquad \bar{\phi}_0^{\mathcal{H}} = \phi_0^{\mathcal{H}} - \phi_0^{I\pm} - \phi_0^{R\pm}. \tag{6.22}$$

Then the stationarity of the following functional

$$\begin{aligned} J(\phi_0^{\mathcal{H}}, \phi_0^{\mathcal{S}}) &= \iint_{\mathcal{H}} \mathrm{d}A \left\{ \frac{1}{2} \left[h(\nabla\phi_0^{\mathcal{H}})^2 - \frac{4\mu^2\Omega^2}{g} \left(\phi_0^{\mathcal{H}}\right)^2 \right] - F_0^{\pm}\phi_0^{\mathcal{H}} \right\} \\ &= \int_{C_o} h \left[(\tfrac{1}{2}\bar{\phi}_0^{\mathcal{S}} - \phi_0^{\mathcal{H}}) \frac{\partial \phi_0^{\mathcal{S}}}{\partial r} - \frac{\bar{\phi}_0^{\mathcal{S}}}{2} \frac{\partial}{\partial r} (\phi_0^{I\pm} + \phi_0^{R\pm}) \right] \mathrm{d}s \end{aligned} \tag{6.23}$$

is equivalent to the corresponding boundary-value problem, as can be easily proved along the lines of Chen and Mei [8], (see also Mei, [20]).

7. Concluding remarks

We have derived the mild-slope approximation which accounts for linear and nonlinear effects to the second order in wave steepness and bed slope for narrow-banded sea waves. The approximate equations include the existing mild-slope equation for the short waves, its second harmonic and its zeroth harmonic with respect to the period of the carrier wave. The zeroth harmonic corresponds to slow oscillations in time and can be used to predict long-period oscillations in a harbor much larger than the length of the wind waves. The long-period waves include nonlinear effects of refraction and diffraction of short waves and have comparable dependence on both short and long spatial scales which cannot be conveniently separated in general.

To develop a theoretical model for practical prediction of harbor oscillations by wind waves, many improvements can be envisioned. If the harbor entrance is much narrower than the typical horizontal dimension of the harbor, significant long-period oscillations occur only after a time much longer than the resonant period. Wind waves from a distant storm may also last from hours to days. Therefore it is desirable to extend the present theory to higher orders in order to account for further nonlinear effects. There are some theoretical efforts to improve the mild-slope equation for the short waves (see Liu, 1990, for a review), but the theoretical framework is not complete unless the long-period part is also extended. Moreover, if the resonant amplification of long-period motion is strong, it may be necessary to replace the present theory by one where the long-period displacement is of the first order, as the short waves. This is a much more laborious undertaking as shown by Foda and Mei [21] for edge-wave resonance. Needless to say, satisfactory representation of wave-breaking of short waves on a gentle beach is essential, but it is very difficult and will likely remain a challenge for a long time.

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References

1. E. C. Bowers, Harbor resonance due to set-down beneath wave groups. *J. Fluid Mech.* 79 (1977) 71–92.
2. C. C. Mei and Y. Agnon, Long-period oscillations in a harbour induced by short waves. *J. Fluid Mech.* 208 (1989) 595–608.
3. J.-K. Wu and P. L.-F. Liu, Harbour excitations by incident wave groups. *J. Fluid Mech.* 217 (1990), 595–613.
4. C. C. Mei and C. Benmoussa, Long waves induced by short-wave groups over anuneven bottom. *J. Fluid Mech.* 139 (1984) 219–235.
5. Y. Agnon, and C. C. Mei, Slow drift motions of a rectangular block in beam seas. *J. Fluid Mech.* 151 (1985) 279–294.
6. Y. Agnon and C. C. Mei, Trapping and resonance of long shelf waves due to groups of short waves. *J. Fluid Mech.* 195 (1988a) 201–221.
7. C. Zhou and P.L.-F. Liu, Second-order low-frequency wave forces on a vertical cylinder. *J. Fluid Mech.* 175 (1987) 143–155.
8. H. S. Chen and C. C. Mei, Oscillations and wave forces in an offshore harbor. Parsons Lab, *MIT Rep*. 190, Dept. Civil and Environm. Engng., MIT, Cambridge (1974) 215 pp.
9. J. R. Houston, Combined refraction and diffraction of short waves using the finite element method. *Appl. Ocean Res.* 3 (1981) 163–170.
10. N. Booij, Gravity waves on water with non-uniform depth and current. *Rep. No. 81-1* (1981) Delft University Technology.
11. A. C. Radder, An explicit Hamiltonian formulation of surface waves in water of finite depth. *J. Fluid Mech.* 237 (1992) 435–455.
12. J. T. Kirby, A general wave equation for waves over rippled beds *J. Fluid Mech.* 162 (1986) 171–186.
13. P. G. Chamberlain and D. Porter, The modified mild-slope equation. *J. Fluid Mech.* (1995) 393–401.
14. D. Porter and D. J. Staziker, Extensions of the mild slope equation. *J. Fluid Mech.* 300 (1995) 367–382.
15. A. Smith and Sprinks, Scattering of surface waves by a conical island. *J. Fluid Mechanics* 72 (1975) 373–384.
16. I. G. Jonsson, *Booij's Current-wave Equation and the Ray Approximation*, Progress Report 54, 15VA, Inst. Hydrodyn. and Hydraulic Engng., Technical University of Denmark (1981) 7–20.
17. J. T. Kirby, Surface wave-current interaction over slowly varying topography. *J. Geophys. Res.* 89 (1984) 745–747.
18. M. W. Dingemans, *Water Wave Propagation Over Uneven Bottoms*. Singapore: World Scientific Publishers (1997) 899 pp.
19. P. L.-F. Liu, Wave transformation in the sea, *Ocean Eng. Sci*. 9 (1990) 27–63.
20. C. C. Mei, *The Applied Dynamics of Ocean Surface Waves*. Singapore: World Scientific (1989) 740 pp.
21. M. A. Foda and C. C. Mei, Nonlinear excitation of long trapped waves by a group of short swells. *J. Fluid Mech.* 111 (1981) 319–345.